ON THE PONDEROMOTIVE FORCES OF INTERACTION OF AN ELECTROMAGNETIC FIELD AND AN ACCELERATING MATERIAL CONTINUUM, TAKING INTO ACCOUNT FINITE DEFORMATIONS

(O PONDEROMOTORNYKH SILAKH VZAIMODEISTVIIA ELEKTROMAGNITNOGO POLIA I USKORENNO DVIZHUSHCHEGOSIA MATERIAL'NOGO KONTINUUMA S UCHETOM KONECHNOSTI DEFORMATSII)

PMM Vol.29, № 1, 1965, pp.4-17

L.I. SEDOV
(Moscow)

(Received November 19, 1964)

The macroscopic motion of a material continuous medium, interacting with an electromagnetic field, is investigated. The results obtained have a universal character, the properties of the medium not being specified in detail. Possible interaction of the moving and deforming medium with an electromagnetic field is taken into account. This interaction is provoked by the presence of electric currents in the medium as well as by the effects of its electrical polarization and magnetization.

Formulas for the ponderomotive force given by various authors differ in nature, and are applicable only to separate, particular cases. This situation is related to the possibility of different definitions of the momentum energy tensor for the electromagnetic field and to complications in the physical problem of the properties of the medium.

To describe the electromagnetic field in the medium, the following electromagnetic quantities are introduced:

$$\mathbf{E}, \quad \mathbf{H}, \quad \mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \quad \mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}, \quad \mathbf{j}, \quad \rho_e \qquad (1)$$

Here, ρ_e is the density of the electric charge distribution, the other quantities are the usual ones [1]. These quantities, valid for an inertial coordinate system, satisfy the (not closed) system of equations of Maxwell [1].

As is well known [2], to write the transformed Maxwell equations for an arbitrary, curvilinear, moving coordinate system, it is convenient to make use of the tensor form of Maxwell's equations, written in four-dimensional form in pseudo-Euclidean Minkowski space. In all of Minkowski space, the metric may be defined by the quadratic form

$$ds^2 = -dx^{1^2} - dx^{2^2} - dx^{3^2} + c^2dt^2 = g_{ij}dx^idx^j \qquad (2)$$

Here, c is the velocity of light.

The set of real values of the variables x^1, x^2, x^3, $x^4 = t$ generate a pseudo-Euclidean space, defined in an inertial Cartesian reference system.

As is well known, every transformation

$$x^i = x^i\,(y^1, y^2, y^3, y^4 = t') \tag{3}$$

for which the equality

$$ds^2 = -\,dy^{1^2} - dy^{2^2} - dy^{3^3} + c^2dt'^2 = g_{ij}dy^idy^j \tag{4}$$

is fulfilled is linear, and is called a Lorentz transformation [3].

The three-dimensional vectors (1) may be defined in an arbitrary inertial coordinate system. To obtain the transformation formulas of the vectors (1) in a four-dimensional Lorentz transformation, it is necessary to introduce two antisymmetric four-dimensional tensors of second oeder, F and H, whose components in inertial Cartesian systems are defined by the matrices (*)

$$F = \| F_{ij} \| = \begin{Vmatrix} 0 & B^3 & -B^2 & cE_1 \\ -B^3 & 0 & B^1 & cE_2 \\ B^2 & -B^1 & 0 & cE_3 \\ -cE_1 & -cE_2 & -cE_3 & 0 \end{Vmatrix}$$

$$H = \| H_{ij} \| = \begin{Vmatrix} 0 & H^3 & -H^2 & cD_1 \\ -H^3 & 0 & H^1 & cD_2 \\ H^2 & -H^1 & 0 & cD_3 \\ -cD_1 & -cD_2 & -cD_3 & 0 \end{Vmatrix} \tag{5}$$

Maxwell's equations may be written in the forms

$$\mathrm{rot}\ \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \quad \mathrm{div}\ \mathbf{B} = 0 \qquad \text{or} \qquad \nabla_l F_{ik} + \nabla_k F_{li} + \nabla_i F_{kl} = 0 \tag{6}$$

$$\mathrm{rot}\ \mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}, \quad \mathrm{div}\ \mathbf{D} = 4\pi\rho_e \qquad \text{or} \qquad \nabla_k H_i^{\ k} = \frac{4\pi}{c} I_i \tag{7}$$

Here $I_1 = -j_1$; $I_2 = -j_2$, $I_3 = -j_3$, $I_4 = \rho_e c^2$ are the covariant components of the four-dimensional electric current vector.

For Lorentz transformations and for arbitrary transformations on the spatial coordinates alone, the vectors **B**, **E** and **H**, **D** can be made to correspond to the antisymmetric tensors F_{ij} and H_{ij}

---

*) Here, and in what follows, in general expressions and in summations, Latin indices $i, j, k, l \ldots$ take values 1,2,3,4, while Greek indices $\alpha, \beta, \gamma, \ldots$ take values 1,2,3.

Furthermore, in operating with three-dimensional of four-dimensional tensors, raising or lowering of indices in accomplished with the help of the metric tensors $g^*_{\alpha\beta}$ or $g_{\alpha\beta}$, defined by the quadratic forms

$$dl^2 = g^*_{\alpha\beta}\, dx^\alpha\, dx^\beta, \qquad ds^2 = g_{ij}\, dx^i\, dx^j = -\,dl^2 + 2g_{4\alpha}\, dx^\alpha\, dt + g_{44}\, dt^2$$

The conversion from the components of these vectors in the x^{α}, t system to the analogous components in the y^{α}, t' system is obtained from general rules for the transformation of the tensor componenets of F_{ij} and H_{ij}. Unlike the vectors **E, D, H, B**, the tensors F and H, their components F_{ij} and H_{ij}, and the tensor equations (6) and (7) have meaning in any noninertial system. Thus, the tensor equations (6) and (7) express invariant physical rules, independent of the choice of coordinate system, which in inertial coordinate systems are Maxwell's equations.

In noninertial coordinate systems, for example, in a coordinate system obtained from a given inertial one by means of a Galilean transformation in the Newtonian sense (without a Lorentz contraction of length and time), the transformed coordinates in the matrices (5) can also be regarded as certain corresponding vectors **B***, **E*** and **H***, **D***. However, these vectors can be regarded as the vectors **B, E** and **H, D** only in an approximate sense for small velocities of a moving system.

To determine the ponderomotive forces, it is necessary to introduce the momentum energy tensor with components $S_{i\cdot}^{\cdot j}$ for the electromagnetic field. The general formulas for the components of the four-dimensional ponderomotive force in an arbitrary coordinate system have the form

$$F_i = -\nabla_k S_i^{\,k} \tag{8}$$

The laws for the variation of the momentum and energy for the system composed of the field plus the material medium may be expressed in the form

$$\nabla_k T_i^{\,k} = F_i + G_i, \qquad \text{or} \qquad \nabla_k \left(T_i^{\,k} + S_{i\cdot}^{\cdot k}\right) = G_i \tag{9}$$

Here, G_i are the components of the four-dimensional external force vector In many cases, it may be assumed that $G_i = 0$. The components of the momentum energy tensor of the medium and the field as one system are represented by the sum

$$I_i^{\,k} = T_{i\cdot}^{\cdot k} + S_{i\cdot}^{\cdot k}$$

In the general case, the momentum energy tensor of the medium, $T_{i\cdot}^{\cdot k}$ characterizes the physical properties and internal interactions in the medium; this tensor also has an electromagnetic nature, since the internal stresses in the material medium are determined either by collisions of particles or direct interaction of atoms and molecules at distances which are large in comparison with the dimensions of the elementary particles of which they are composed. As is well known, in both cases these microscopic interactions are of an electromagnetic nature. According to the appropriate definition of the model of the continuous medium, the components $T_{i\cdot}^{\cdot k}$ are connected with the metric tensor, with the four-dimensional velocity vector of points of the medium, with the thermodynamic functions of state, and with the characteristics of the dissipative mechanisms in the medium (*).

---

*) The tensor T and its components $T_{i\cdot}^{\cdot k}$ can be regarded as functions of constant and variable tensors and scalar parameters, defining the structure, physical state, and internal processes for infinitesimal particles.

The decomposition of the total momentum energy tensor $I_i^{\cdot k}$ into the sum $T_{i\cdot}^{\cdot k} + S_{i\cdot}^{\cdot k}$ for the material medium and the field is directly connected with the decomposition of the total electromagnetic force acting on a small portion of the medium – a body force and a surface force. The internal surface stresses in the medium are defined by the components of the tensor $T_{\alpha\cdot}^{\cdot\beta}$, and the electromagnetic body forces by the components of the vector $F_\alpha = -\nabla_k S_{\alpha\cdot}^{\cdot k}$.

It is evident that, for a unique determination of the tensor $I_{i\cdot}^{\cdot k}$, which is essential on physical grounds, the tensors $T_{i\cdot}^{\cdot k}$ and $S_{i\cdot}^{\cdot k}$ may be determined differently, and this is essentially connected with the different methods for decomposing a single electromagnetic system into two interacting electromagnetic systems.

It is essential that after choosing $S_{i\cdot}^{\cdot k}$ for the field, the tensor $T_{i\cdot}^{\cdot k}$ for the material medium be defined with this choice of $S_{i\cdot}^{\cdot k}$ taken into account in a unique way.

From what has been said above, it is evident that it is possible to have the well-known arbitrariness in the specification of $S_{i\cdot}^{\cdot k}$; this circumstance has been the basis of many discussions and for the derivation of a variety of formulas by different authors [1 and 3 to 5] for the ponderomotive forces, this question frequently being considered quite without regard to the choice of the tensor $T_{i\cdot}^{\cdot k}$ for the material medium.

Let us investigate the formula for the ponderomotive force when the tensor $S_{i\cdot}^{\cdot k}$ in an arbitrary coordinate system is defined by Minkowski's tensor formula

$$S_{i\cdot}^{\cdot k} = -\frac{1}{4\pi}\left[F_{mi}H^{mk} - \frac{1}{4}\delta_{i\cdot}^{\cdot k}F_{mn}H^{mn}\right] \tag{10}$$

Minkowski's tensor is not symmetrical in the general case, i.e.

$$S_{ij} \neq S_{ji}$$

Making use of Equation (10) and the conditions of antisymmetry for F_{ij} and H_{ij}, we obtain on the basis of Equations (6) and (7) (*)

$$F_i = \frac{1}{c}F_{mi}I^m - \frac{1}{16\pi}[F_{mk}\nabla_i H^{mk} - H^{mk}\nabla_i F_{mk}] \tag{11}$$

The tensor equations (6), (7) and Formulas (10) and (11) are valid in any moving and general curvilinear coordinate system.

Together with the tensors F and H, we can introduce another antisymmetric tensor P, defined by

$$P = \frac{1}{4\pi}(F - H), \qquad \|P_{ij}\| = \left\| \begin{matrix} 0 & M^3 & -M^2 & -cP_1 \\ -M^3 & 0 & M^1 & -cP_2 \\ M^2 & -M^1 & 0 & -cP_3 \\ cP_1 & cP_2 & cP_3 & 0 \end{matrix} \right\| \tag{12}$$

*) In deriving (11), use was made of the fact, on the basis of (6),

$$H^{mk}\nabla_k F_{mi} = \frac{1}{2}H^{mk}\nabla_i F_{mk}$$

In an inertial coordinate system, the tensor $\mathbf{P}$ is formed with the help of the three-dimensional electric polarization vector $\mathbf{P}\,(P_1, P_2, P_3)$ and the magnetization vector $\mathbf{M}\,(M^1, M^2, M^3)$. With the help of the tensor P, Equation (11) may be written in the form

$$F_i = \frac{1}{c} F_{mi} I^m + \frac{1}{4} [F_{mk} \nabla_i P^{mk} - P^{mk} \nabla_i F_{mk}] \tag{13}$$

The first term in Equations (11) and (13) defines the Lorentz force, the second term becomes zero in the absence of polarization or magnetization, the symbols ∇_i denote covariant, four-dimensional derivatives.

If the reference system is inertial, then in connection with the tensors F and P we can introduce a system of three-dimensional vectors (1). In an inertial coordinate system, Equation (13) may be written in the form

$$F^\alpha = \rho_e E_\alpha + \frac{1}{c} [\mathbf{j} \times \mathbf{B}]_\alpha + \frac{1}{2} \left[\mathbf{P} \frac{\partial \mathbf{E}}{\partial x^\alpha} - \mathbf{E} \frac{\partial \mathbf{P}}{\partial x^\alpha} + \mathbf{M} \frac{\partial \mathbf{B}}{\partial x^\alpha} - \mathbf{B} \frac{\partial \mathbf{M}}{\partial x^\alpha} \right] \tag{14}$$

where the four-dimensional contravariant force components F^α correspond to spatial three-dimensional covariant force components. Equation (14) preserves its form in going from a Cartesian to a curvilinear spatial coordinate system.

From Equation (13), in an inertial coordinate system, we obtain (*) for $i = 4$,

$$F^4 c^2 = F_4 = \mathbf{E} \cdot \mathbf{j} + E_\beta \frac{\partial P^\beta}{\partial t} + B_\beta \frac{\partial M^\beta}{\partial t} - \frac{\partial}{\partial t} \frac{E_\beta P^\beta + B_\beta M^\beta}{2} \tag{15}$$

Equations (14) and (15) are directly useful for defining the ponderomotive forces by means of the vector system (1) when the material medium is at rest or in a state of inertial translational motion. In the latter case, if the vectors $\mathbf{E}$, $\mathbf{P}$, $\mathbf{B}$ and $\mathbf{M}$ are defined in an inertial reference system tied to the body, then in Equation (15) the term $\mathbf{E} \cdot \mathbf{j}$ gives the Joule heat, the term $E_\beta \partial P^\beta / \partial t + B_\beta \partial M^\beta / \partial t$ may be considered to be the macroscopic inflow of energy from the field to the body, due to the microscopic mechanisms of polarization and magnetization of the body, while the quantity $^1/_2\,(E_\beta P^\beta + B_\beta M^\beta)$ is conveniently included in the internal energy of the material medium.

It is easy to see that, in Equations (14) in the second term and everywhere in (15), the components of the vector $\mathbf{B}$ can be replaced by the components of the vector $\mathbf{H}$.

If the body is accelerating and is deforming, then Equation (13), which is valid in any coordinate system, may be used; in particular, this is true also for a Lagrangian coordinate system moving with the body, in which the

---

*) Equation (15) is obtained as a consequence of the vector identity

$$\Pi = S_{4\cdot}^{\cdot\alpha} \mathbf{a}_\alpha = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H}$$

which is valid for various definitions of the momentum energy tensor for the field, in particular, for that of Minkowski as well as that of Abraham.

three-dimensional velocities of all points of the medium are always equal to zero (*).

Within special relativity theory, space-time forms a four-dimensional, pseudo-Euclidean space; therefore, we can take as inertial reference system K a Cartesian coordinate system $x^1, x^2, x^3, x^4 = t$, for which the metric is given by Equation (2).

In a coordinate system $L(\xi^1, \xi^2, \xi^3, \xi^4 = t^\wedge)$ moving with the body, we have for the metric

$$ds^2 = g^\wedge{}_{ij} d\xi^i d\xi^j, \qquad g^\wedge{}_{ij}(\xi^k) \tag{16}$$

Since the coordinates x^i and ξ^i are taken in one and the same pseudo-Euclidean space, there exist functional relations (**)

$$x^i = x^i(\xi^1, \xi^2, \xi^3, t^\wedge) \tag{17}$$

which define the law of motion in the continuum under consideration.

For the components of the four-dimensional velocity, we have the following in system K (18)

$$u^\alpha = \left(\frac{dx^\alpha}{ds}\right)_{\xi^\alpha} = \left(\frac{\partial x^\alpha}{\partial t}\right)_{\xi^\alpha}\left(\frac{\partial t}{\partial s}\right)_{\xi^\alpha}, \qquad u^4 = \left(\frac{\partial t}{\partial s}\right)_{\xi^\alpha} = \frac{1}{\sqrt{c^2 - v^2}} \qquad \left(v^2 = \sum_\alpha \left(\frac{\partial x^\alpha}{\partial t}\right)^2\right)$$

in system L

$$u^{\wedge\alpha} = \frac{d\xi^\alpha}{ds} = 0, \qquad u^{\wedge 4} = \frac{dt^\wedge}{ds} = \frac{1}{\sqrt{g^\wedge{}_{44}}} \tag{19}$$

With the help of (17) and (18), we can write for the components of $g^\wedge_{ij}$ Formulas

$$\begin{aligned}
g^\wedge{}_{\alpha\beta} &= -\sum_{\gamma=1}^{3} \frac{\partial x^\gamma}{\partial \xi^\alpha}\frac{\partial x^\gamma}{\partial \xi^\beta} + c^2\left(\frac{\partial t}{\partial \xi^\alpha}\right)_{t^\wedge}\left(\frac{\partial t}{\partial \xi^\beta}\right)_{t^\wedge} \\
g^\wedge{}_{\alpha 4} = g^\wedge{}_{4\alpha} &= \left[-\sum_{\gamma=1}^{3} \frac{\partial x^\gamma}{\partial \xi^\alpha}\frac{\partial x^\gamma}{\partial t} + c^2 \frac{\partial t}{\partial \xi^\alpha}\right]\left(\frac{\partial t}{\partial t^\wedge}\right)_{\xi^\alpha} \\
g^\wedge{}_{44} &= -\sum_{\gamma=1}^{3}\left(\frac{\partial x^\gamma}{\partial t^\wedge}\right)^2 + c^2\left(\frac{\partial t}{\partial t^\wedge}\right)^2_{\xi^\alpha} = (c^2 - v^2)\left(\frac{\partial t}{\partial t^\wedge}\right)_{\xi^\alpha}
\end{aligned} \tag{20}$$

For every point $M(\xi^1, \xi^2, \xi^3)$ of the moving continuum L at every moment of time t, it is possible to choose a proper inertial coordinate system K, so that the three-dimensional velocity $\mathbf{v}$ of the point M in system K be equal to zero. In system K, the acceleration of the point M and the velocity of neighboring points of system L are different from zero, in general.

To simplify the derivation of the invariant scalar or tensor relations, use can be made of the freedom in the choice of system L at a given moment

---

*) Specification of the three-dimensional velocity field as a function of time in some fixed coordinate system makes it possible to individualize points of the continuum and thus to introduce a Lagrangian coordinate system.

**) According to (17), the relation between x^α, t and $\xi^\alpha, t^\wedge$ is reciprocally unique in finite space.

of time $t^{\wedge *}$ and the proper system K for the point M at the moment $t^{\wedge *}$.
$t^{\wedge *}$

Along with the coordinates ξ^α and $t^\wedge$, we can introduce the Lagrangian variables $\eta^\alpha, \eta^4 = t'^\wedge$, (in general nonholonomic), determined by relations of the form

$$(A) \quad d\eta^\alpha = A^\alpha{}_\beta (\xi^\alpha, t^\wedge)\, d\xi^\beta, \qquad dt'^\wedge = B_\beta (\xi^\alpha, t^\wedge)\, d\xi^\beta + B(\xi^\alpha, t^\wedge)\, dt^\wedge$$

where $A_\beta{}^\alpha$ and B_β are components of some tensor and vector, for which $|A^\alpha_{.\beta}| \neq 0$ and $B \neq 0$. In the case of holonomic coordinates, the conditions of integrability are fulfilled, and therefore relations (A) reduce to finite relations

$$(B) \quad \eta^\alpha = \eta^\alpha (\xi^1, \xi^2, \xi^3), \qquad t'^\wedge = t^\wedge (\xi^\alpha, t^\wedge)$$

Corresponding to the vector base $\mathfrak{s}_\alpha{}^\wedge$, $\mathfrak{s}_4{}^\wedge$ in the variables ξ^k is the vector base $\mathfrak{s}_\alpha{}'^\wedge$, $\mathfrak{s}_4{}'^\wedge$ in the quasi-coordinates η^k. It is evident that these bases are related by Formulas

$$\mathfrak{s}_\beta{}^\wedge = A_\beta{}^\alpha \mathfrak{s}_\alpha{}'^\wedge + B_\beta \mathfrak{s}_4{}'^\wedge, \qquad \mathfrak{s}_4{}^\wedge = B \mathfrak{s}_4{}'^\wedge$$

If $t^\wedge$ and $t'^\wedge$ are defined as proper times in systems ξ^k and η^k, respectively, then the second of relations (B) has the form

$$(C) \qquad t'^\wedge = t^\wedge + f(\xi^1, \xi^2, \xi^3) \quad \text{or} \quad B = 1$$

The function $f(\xi^\alpha)$ gives the reference origin for the time on different world lines; if f = const , then $B_\beta = 0$.

In nonholonomic coordinates, relations (C) are replaced by Equation

$$dt'^\wedge = dt^\wedge + B_\beta d\xi^\beta$$

In both holonomic and nonholonomic cases, the quadratic form $\overset{*}{g}_{\alpha\beta} d\eta^\alpha d\eta^\beta$ defines a three-dimensional metric, which is non-Euclidean in general.

The choice of L and K may be made so that

$$x^\alpha = \xi^\alpha \quad \text{for} \quad t^\wedge = t^{\wedge *}, \qquad \sqrt{1 - \frac{v^2}{c^2}}\, dt = dt^\wedge \tag{21}$$

This means that, for $t^\wedge = t^{\wedge *}$, in all three-dimensional space, the spatial coordinates ξ^α coincide with Cartesian coordinates x^α in K , and that dt is the increment of proper time in K , while $dt^\wedge$ is the infinitesimal increment of proper time at points of system L . At point M we have $v = 0$ and, consequently, the increments of proper time in K and in L are identical for point M .

From Equations (21) it follows that in all three-dimensional space the equations

$$\frac{\partial x^\alpha}{\partial \xi^\beta} = \delta_\beta{}^\alpha, \qquad \left(\frac{\partial t}{\partial \xi^\alpha}\right)_{t^\wedge} = 0, \qquad \left(\frac{\partial x^\alpha}{\partial t}\right)_{\xi^\alpha} = v^\alpha, \qquad \left(\frac{\partial t}{\partial t^\wedge}\right)_{\xi^\alpha} = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{22}$$

are valid for $t^\wedge = t^{\wedge *}$, and in addition, at point M , the following equations are valid:

$$\frac{\partial x^\alpha}{\partial t} = 0, \qquad \left(\frac{\partial t}{\partial t^\wedge}\right)_{\xi^\alpha} = 1, \qquad \frac{\partial^2 x}{\partial \xi^\beta \partial \xi^\gamma} = 0$$

$$\frac{\partial^2 x^\alpha}{\partial t^\wedge \partial \xi^\beta} = \frac{\partial^2 x^\alpha}{\partial t \partial \xi^\beta} = \frac{\partial v^\alpha}{\partial \xi^\beta} = \frac{\partial v_\alpha}{\partial \xi^\beta}, \qquad \frac{\partial^2 x^\alpha}{\partial t^{\wedge 2}} = \frac{\partial^2 x^\alpha}{\partial t^2} = \frac{\partial v^\alpha}{\partial t^\wedge} \tag{23}$$

since

$$\left(\frac{\partial^2 t}{\partial t^{\wedge 2}}\right)_{\xi^\alpha} = \frac{\partial}{\partial t^\wedge} \frac{1}{\sqrt{1 - v^2/c^2}} = 0, \qquad \frac{\partial^2 t}{\partial t^\wedge \partial \xi^\alpha} = 0$$

$v^\alpha = v_\alpha$ denotes components in system K of the three-dimensional velocity vector of a point of the medium.

From (20) and (22) it follows that, for $t^\wedge = t^{\wedge *}$, Equations

$$g^\wedge{}_{\alpha\beta} = \begin{cases} -1 & (\alpha = \beta) \\ 0 & (\alpha \neq \beta) \end{cases}, \qquad g^\wedge{}_{4\alpha} = -\frac{v_\alpha}{\sqrt{1 - v^2/c^2}}, \qquad g^\wedge{}_{44} = c^2 \tag{24}$$

are valid, with $g^\wedge{}_{4\alpha} = 0$ at point M.

Spatial coordinates in the inertial system K and in the noninertial system L coincide only for $t^\wedge = t^{\wedge *}$; at subsequent times, accelerations and deformations cause coordinate lines in system L to displace and deform relative to system K.

We shall denote by $\mathfrak{z}_i$, $\mathfrak{z}^i$ and $\mathfrak{z}^\wedge{}_i$, $\mathfrak{z}^{\wedge i}$, respectively, the covariant and contravariant base vectors in systems K and L.

Let us consider an arbitrary tensor field N.

$$N = N^{ijk}_{\cdots}\, \mathfrak{z}_i \mathfrak{z}_j \mathfrak{z}^k \ldots = N^{\wedge ij}{}_{k\cdots} \mathfrak{z}^\wedge{}_i \mathfrak{z}^\wedge{}_j \mathfrak{z}^{\wedge k} \ldots$$

and the gradient of the tensor N.

$$\operatorname{grad} N = \frac{\partial N}{\partial x^l} \mathfrak{z}^l = \frac{\partial N}{\partial \xi^l} \mathfrak{z}^{\wedge l} = \nabla^\wedge{}_l N^{ij}{}_{k\ldots} \mathfrak{z}^\wedge{}_i \mathfrak{z}^\wedge{}_j \mathfrak{z}^{\wedge k} \ldots \mathfrak{z}^{\wedge l}$$

For $t^\wedge = t^{\wedge *}$, due to conditions (22) and (23), we have at point M Equations $\mathfrak{z}_i = \mathfrak{z}^\wedge{}_i$ and $\mathfrak{z}^i = \mathfrak{z}^{\wedge i}$, therefore

$$N^{ij}{}_{k\ldots} = N^{\wedge ij}{}_{k\ldots}, \qquad \frac{\partial N^{ij}{}_{k\ldots}}{\partial x^l} = \nabla_l{}^\wedge N^{\wedge ij}{}_{k\ldots} \tag{25}$$

In going from point M to other points, or for $t^\wedge \neq t^{\wedge *}$, Equations (25) may be violated.

In comparing various physical equations in their components, use can be made of the coordinate systems L and K. In equations containing derivatives of components of tensors with respect to coordinates or to time, use can be made of Equation (25), and the derivatives for different components in different systems applied, depending in which system the components under consideration are specified.

In mumber of cases, it is convenient to specify and consider the components of the momentum energy tensor of the material medium in a co-moving coordinate system L, while at the same time giving the components of the momentum energy tensor $S_{i\cdot}^{\cdot k}$ of the electromagnetic field in an inertial coordinate system K,

In making use of the proper system K, the three-dimensional vector characteristics of the electromagnetic field and the corresponding equations of Maxwell in vector form can be introduced. At the same time, the three-dimensional vectors introduced for system K can be considered in the spatial coordinates of L. Thus, the system K may be regarded as a supplementary method for determining the ordinary vector characteristics of the electromagnetic field. If, for the electromagnetic field, we limit ourselves to the tensors F, H, P and S, then all the tensors may be investigated only in a co-moving coordinate system. In this case, introduction of the inertial system K may be essential for determining the components of the tensor $g^\wedge_{ij}$ (Equations (20)) and of the four-dimensional velocity vector. Both methods are essential, generally speaking, for determining the momentum energy tensors of the electromagnetic field and the material medium.

For the ponderomotive forces and for the inflow of energy in the general case of motion of a deformable medium, use can be made of Equations (14) and (15), in which the components of the vectors $E^{\alpha}, B^{\alpha}, P^{\alpha}, M^{\alpha}$ are defined in a spatial coordinate system of the inertial system K. In Equations (14) and (15), coordinate lines in system K may be taken to be curvilinear. In using the vectors $E^{\wedge\alpha}\hat{\partial}_{\alpha}$, $P^{\wedge\alpha}\hat{\partial}_{\alpha}$, $B^{\wedge\alpha}\hat{\partial}_{\alpha}$, $M^{\wedge\alpha}\hat{\partial}_{\alpha}$ in a co-moving coordinate system L, Equations (14) preserve their form (*).

If the quantities $E^{\wedge\alpha}, P^{\wedge\alpha}, B^{\wedge\alpha}, M^{\wedge\alpha}$ are introduced in Equation (15), then it is necessary to take into account Equations [6]

$$\left(\frac{\partial P^{\alpha}}{\partial t}\right)_{x^{\alpha}} = \left(\frac{\partial P^{\wedge\alpha}}{\partial t^{\wedge}}\right)_{\xi^{\alpha}} + P^{\wedge\beta}\left(e^{\alpha\cdot}_{\cdot\beta} + \omega^{\alpha\cdot}_{\cdot\beta}\right)$$

$$\left(\frac{\partial M^{\alpha}}{\partial t}\right)_{x^{\alpha}} = \left(\frac{\partial M^{\wedge\alpha}}{\partial t^{\wedge}}\right)_{\xi^{\alpha}} + M^{\wedge\beta}\left(e^{\alpha\cdot}_{\cdot\beta} + \omega^{\alpha\cdot}_{\cdot\beta}\right) \tag{26}$$

Here $e^{\wedge}{}_{\alpha\beta}$ and $\omega^{\wedge}{}_{\alpha\beta} = {}^{1}/_{2}\left(\partial v_{\alpha}/\partial\xi^{\beta} - \partial v_{\beta}/\partial\xi^{\alpha}\right)$ are the components of the three-dimensional rate of deformation and vorticity tensors, defined for a three-dimensional velocity vector $\mathbf{v}$ of points travelling with system L relative to the corresponding proper system K.

On the basis of (26), Equation (16) takes the form

$$F_4 = -\nabla_k S^{\cdot k}_{4\cdot} = \mathbf{E}\cdot\mathbf{j} + \left(E^{\wedge\alpha}P^{\wedge\beta} + B^{\wedge\alpha}M^{\wedge\beta}\right)\left(e^{\wedge}{}_{\alpha\beta} + \omega^{\wedge}{}_{\alpha\beta}\right) +$$

$$+ E^{\wedge}{}_{\beta}\frac{\partial P^{\wedge\beta}}{\partial t^{\wedge}} + B^{\wedge}{}_{\beta}\frac{\partial M^{\wedge\beta}}{\partial t^{\wedge}} - \frac{\partial}{\partial t^{\wedge}}\frac{E^{\wedge}{}_{\beta}P^{\wedge\beta} + B^{\wedge}{}_{\beta}M^{\wedge\beta}}{2} \tag{27}$$

The scalar energy equation for the system consisting of the material medium and the field in an arbitrary coordinate system may be written in the form

$$u^{i}\nabla_k T^{\cdot k}_{i\cdot} + u^{i}\nabla_k S^{\cdot k}_{i\cdot} = u^{i}G_i = d^{*}q/c\,dt^{\wedge} \tag{28}$$

Here, $d^{*}q/c\,dt^{\wedge}$ is the external, macroscopic inflow of energy into unit area per unit proper time, due to interactions with other bodies not included in the tensors $T^{\cdot k}_{i\cdot}$ and $S^{\cdot k}_{i\cdot}$; in many cases we can usually assume that $d^{*}q/dt^{\wedge} = 0$.

In Equation (28), in accordance with (19), (24) and (25), the term $\nabla^{\wedge}{}_{\alpha}T^{\wedge\alpha}_{4}$ is taken in a co-moving coordinate system L, and the terms $\nabla_k S^{\cdot k}_{4\cdot}$ and $\nabla_4^{\wedge}T_4^{\wedge 4} = (\partial T_4^{4}/\partial t^{\wedge})_{x^{\alpha}} = (\partial T_4^{4}/\partial t^{\wedge})_{\xi^{\alpha}}$ in the inertial, proper coordinate system K; thus we obtain

$$\frac{\partial T_4^{4}}{\partial t} + \nabla^{\wedge}{}_{\alpha}T_4^{\wedge\alpha} = F_4 + d^{*}q/dt^{\wedge} \tag{29}$$

where F_4 is defined by Equation (27).

---

*) This derivation follows from the equations for transforming the vectors (1) when the system K is introduced at each point of the medium and from Equations $\Gamma^{\wedge k}_{\beta\gamma} = 0$. Equation (14) is preserved also for transformations of the form $\eta^{\alpha} = \eta^{\alpha}(\xi^1, \xi^2, \xi^3)$ and $t'^{\wedge} = t^{\wedge}$.

To transform and evaluate the left-hand side of (29), we note that, at the point M, for which $v^{\alpha}=0$ at $t^{\wedge}=t^{\wedge *}$, Equations (20), (22) and (23) result in Equations

$$\frac{\partial g^{\wedge}{}_{\beta\gamma}}{\partial \xi^{\alpha}}=0, \qquad \frac{\partial g^{\wedge}{}_{4\beta}}{\partial \xi^{\alpha}}=-\frac{\partial v_{\beta}}{\partial \xi^{\alpha}}, \qquad \frac{\partial g^{\wedge}{}_{44}}{\partial \xi^{\alpha}}=0 \tag{30}$$

$$\frac{\partial g^{\wedge}{}_{\alpha\beta}}{\partial t^{\wedge}}=-\left(\frac{\partial v_{\beta}}{\partial \xi^{\alpha}}+\frac{\partial v_{\alpha}}{\partial \xi^{\beta}}\right), \qquad \frac{\partial g^{\wedge}{}_{\alpha 4}}{\partial t^{\wedge}}=-\frac{\partial v^{\alpha}}{\partial t^{\wedge}}, \qquad \frac{\partial g^{\wedge}{}_{44}}{\partial t^{\wedge}}=0$$

Using (30), we find that, for Cheistoffel symbols,

$$\Gamma^{\wedge}{}^{i}_{jk}=\frac{1}{2}g^{\wedge il}\left[\frac{\partial g^{\wedge}{}_{jl}}{\partial \xi^{k}}+\frac{\partial g^{\wedge}{}_{kl}}{\partial \xi^{j}}-\frac{\partial g^{\wedge}{}_{jk}}{\partial \xi^{l}}\right]$$

Equations

$$\Gamma^{\wedge}{}^{k}_{\beta\gamma}=\Gamma^{\wedge}{}^{4}_{k4}=\Gamma^{\wedge}{}^{4}_{4k}=0, \qquad \Gamma^{\wedge}{}^{\alpha}_{4\beta}=\frac{\partial v^{\alpha}}{\partial \xi^{\beta}}=\frac{\partial v_{\alpha}}{\partial \xi^{\beta}}, \qquad \Gamma^{\wedge}{}^{\beta}_{44}=\frac{\partial v^{\beta}}{\partial t^{\wedge}} \tag{31}$$

are valid. On the basis of (31), we have

$$\nabla^{\wedge}{}_{\alpha}T^{\wedge}{}_{4}^{\alpha}=\frac{\partial T^{\wedge\alpha}_{4}}{\partial \xi^{\alpha}}+T^{\wedge i}_{4}\Gamma^{\wedge}{}^{\alpha}_{i\alpha}-T^{\wedge}{}_{j}^{\alpha}\Gamma^{\wedge}{}^{j}_{\alpha 4}=\operatorname{div}\mathbf{Q}+T^{\wedge}{}^{\cdot 4}_{4\cdot}\operatorname{div}\mathbf{v}-p^{\alpha\beta}\frac{\partial v^{\wedge}{}_{\alpha}}{\partial \xi^{\beta}}$$

$$(\mathbf{Q}=T^{\wedge\alpha}_{4}\mathfrak{s}^{\wedge}{}_{\alpha},\ p^{\alpha\beta}=-T^{\wedge\alpha\beta}) \tag{32}$$

Further rearrangement of Equation (2.9) can be accomplished by using Equation

$$\frac{d\rho}{dt^{\wedge}}+\rho\operatorname{div}\mathbf{v}=0 \tag{33}$$

where ρ is the density, determined in the co-moving coordinate system from the relation $\rho\, d\tau_0=dm_0$ $\qquad (d\tau_0=\sqrt{g^{*}}\,d\xi_1\,d\xi_2\,d\xi_3,\ g^{*}(\xi^{\alpha},t^{\wedge})=|g^{*}_{\alpha\beta}|)$

Here, $d\tau_0$ is the substantial volume in the co-moving system, dm_0 is the rest mass, $g^{*}_{\alpha\beta}$ are the components of the three-dimensional metric tensor.

We now define

$$T^{\cdot 4}_{4\cdot}+\frac{1}{2}(E_{\beta}P^{\beta}+B_{\beta}M^{\beta})=\rho U \tag{34}$$

With (27), (32), (33) and (34), we can write (29) in the form

$$\left(\frac{\partial U}{\partial t^{\wedge}}\right)_{\xi^{\alpha}}=\left[\frac{p^{\alpha\beta}}{\rho}-\frac{1}{2}(E_{\gamma}\pi^{\gamma}+B_{\gamma}m^{\gamma})g^{*\alpha\beta}+E^{\wedge\alpha}\pi^{\wedge\beta}+B^{\wedge\alpha}m^{\wedge\beta}\right]\nabla_{\beta}v_{\alpha}+$$

$$+E^{\wedge}{}_{\beta}\frac{\partial \pi^{\wedge\beta}}{dt^{\wedge}}+B^{\wedge}{}_{\beta}\frac{\partial m^{\wedge\beta}}{\partial t^{\wedge}}+\frac{1}{\rho}\,\mathbf{E}.\mathbf{j}-\frac{1}{\rho}\operatorname{div}\mathbf{Q}+\frac{1}{\rho}(d^{*}q/dt^{\wedge}) \tag{35}$$

where $\pi^{\alpha}=P^{\alpha}/\rho, m^{\alpha}=M^{\alpha}/\rho$, are the components of the three-dimensional vectors of the moments of polarization and magnetization, per unit rest mass.

Equation (35) is called the equation of heat influx, and is valid for irreversible as well as reversible processes. The quantity U can be regarded, in a co-moving reference system, as the local specific internal energy per unit rest mass of the material medium. In the general case, the full internal energy of finite volumes of the medium, due to internal macroscopic interactions in the material medium, cannot be represented in the form of an integral of U. The specific internal energy U, the specific entropy S and the absolute temperature T (*), defined above in a co-moving coordinate system, may be regarded as scalar quantities, just as $dm_0 = \rho d\tau_0$. Together with the quantity U, it is convenient to use the specific free energy F, defined by Equation $F = U - TS$.

With the help of the function F, Equation (35) can be rewritten in the form

$$(dF)_{\xi^\alpha} = -S\,dT + \left[\frac{p^{\alpha\beta}}{\rho} - \frac{1}{2}(E_\gamma \pi^\gamma + B_\gamma m^\gamma)\,g^{*\alpha\beta} + \right.$$
$$\left. + E^\alpha \pi^\beta + B^\alpha m^\beta\right](e_{\alpha\beta} + \omega_{\alpha\beta})\,dt^\wedge +$$
$$+ E^\wedge{}_\alpha\, d\pi^{\wedge\alpha} + B^\wedge{}_\alpha\, dm^{\wedge\alpha} - \frac{1}{\rho}\operatorname{div} Q\, dt^\wedge + \frac{1}{\rho}\mathbf{E}.\,\mathbf{j}\,dt^\wedge + \frac{1}{\rho}d^*q - T\,dS \quad (36)$$

In what follows, we shall consider all the quantities appearing in the heat influx equation (36) as three-dimensional scalars,vectors, and tensors.

The vectors $\mathbf{E}$ and $\mathbf{j}$ are taken in the proper coordinate system K, and therefore the energy flux $\rho^{-1}\mathbf{E}\cdot\mathbf{j}$ represents the Joule heat.

The energy flux $-\rho^{-1}\operatorname{div} \mathbf{Q}dt^\wedge$ can be represented in the form of a sum of inflow of heat energy and nonheat energy; this inflow is expressed as the flux of the vector $\mathbf{Q}dt^\wedge = T^{\wedge\cdot\alpha}_{4\cdot}\mathfrak{d}^\wedge{}_\alpha\, dt^\wedge$ on the boundary of a small particle. It is evident that the vector $\mathbf{Q}$, just as the components $T^{\wedge\cdot 4}_{4\cdot}$, can depend only on the same defining quantities as the momentum energy tensor $T^{\wedge\cdot j}_{i\cdot}$.

Energy flux which is independent of the momentum energy tensor, for example, radiant energy flux, is included in the term $\rho^{-1}d^*q$.

Equation (36) is satisfied for all possible processes in the medium occurring due to the action of arbitrary external forces, for arbitrary changes of the determining parameters. Making use of this, Equation (36) may be used as the basis for deriving the equations of state and the kinetic equations which are satisfied for any process. These physical relations can be obtained when the free energy F and the entropy increment $dS = d_e S + d_i S$ are given as functions of the specified quantities ($d_e S$ is the inflow of entropy across the surface bounding the volume of a small particle).

In constructing specific models of material media, it is quite consistent to assume the absence of connecting relations between geometric or kinematic quantities, differential or any other relations, different from their direct definition. An example of such a connection could be the condition of incompressibility, which, however, can be applied in some cases. The existence of supplementary relations leads to restrictions in the laws of motion which are independent of external conditions or of the effect of external body or surface forces on the boundaries of finite volumes or small particles of the medium.

We shall investigate Equation (36) under the assumption that the free energy F can be considered to be a function of the following parameters:

$$T,\quad g^\circ{}_{\alpha\beta},\quad g^\wedge{}_{\alpha\beta},\quad \pi^{\wedge\alpha},\quad m^{\wedge\alpha},\quad \nabla^\wedge{}_\beta \pi^{\wedge\alpha},\quad \nabla^\wedge{}_\beta m^{\wedge\alpha},\quad \frac{\partial g^\wedge{}_{\alpha\gamma}}{\partial \xi^\beta},\quad \frac{\partial g^0_{\alpha\gamma}}{\partial \xi^\beta} \quad (37)$$

---

*) In what follows, we shall consider reversible processes or only those irreversible processes for which the concept of temperature and free energy is meaningful.

where $g^{\circ}_{\alpha\beta}$ is the three-dimensional metric tensor of some initial state. Since $\nabla^{\wedge}_{\beta} g^{\wedge}_{\alpha\gamma} = 0$, and

$$\nabla^{\wedge}_{\beta} g^{\circ}_{\alpha\gamma} = \frac{\partial g^{\circ}_{\alpha\gamma}}{\partial \xi^{\beta}} - g^{\circ}_{\alpha\mu}\Gamma^{\wedge\mu}_{\beta\gamma} - g^{\circ}_{\mu\gamma}\Gamma^{\wedge\mu}_{\alpha\beta}$$

therefore, in the arguments of the function F, we can take $\partial g^{\wedge}_{\alpha\gamma} / \partial\xi^{\beta}$ as variable quantities with time and $g^{\circ}_{\alpha\beta}$ and $\partial g^{\circ}_{\alpha\gamma} / \partial \xi^{\beta}$ as constants. We assume, in addition,

$$Q^{\beta} dt^{\wedge} = R^{\wedge\cdot\beta}_{\alpha\cdot} d\pi^{\wedge\alpha} + N^{\wedge\cdot\beta}_{\alpha\cdot} dm^{\wedge\alpha} + \Lambda^{\wedge\beta\alpha\gamma} dg^{\wedge}_{\alpha\gamma} + \Omega^{\wedge\beta} dt^{\wedge} \tag{38}$$

where the coefficients $R^{\wedge\cdot\beta}_{\alpha\cdot}$, $N^{\wedge\cdot\beta}_{\alpha\cdot}$, $\Lambda^{\wedge\beta\alpha\gamma}$ and $\Omega^{\wedge\beta}$ depend on the parameters (37) and, in the general case, on certain other quantities (*).

We can supplement system (37) with other parameters and include certain derivatives with respect to time. In these, more general cases, the development of a subsequent theory is also possible, with complications.

It is not difficult to verify Equations

$$d\nabla_{\beta}\,\pi^{\alpha} = \nabla_{\beta} d\pi^{\alpha} + \pi^{\gamma} d\Gamma^{\alpha}_{\gamma\beta}$$

where

$$d\Gamma^{\alpha}_{\gamma\beta} = -\Gamma^{\lambda}_{\gamma\beta} g^{\alpha\mu}\, dg_{\lambda\mu} + \frac{1}{2} g^{\alpha\mu}\left(d\,\frac{\partial g_{\gamma\mu}}{\partial\xi^{\beta}} + d\,\frac{\partial g_{\beta\mu}}{\partial\xi^{\gamma}} - d\,\frac{\partial g_{\gamma\beta}}{\partial\xi^{\mu}}\right) \tag{39}$$

$$\nabla_{\beta}\, dg_{\alpha\gamma} = d\,\frac{\partial g_{\alpha\gamma}}{\partial\xi^{\beta}} - dg_{\alpha\mu}\Gamma^{\mu}_{\gamma\beta} - dg_{\mu\gamma}\Gamma^{\mu}_{\alpha\beta}$$

With Equations (37) to (39), Equation (36) can be written in the form

$$\begin{aligned}\varphi\, dT + \psi^{\alpha\beta} e_{\alpha\beta}\, dt^{\wedge} + \Omega^{\alpha\beta}\omega_{\alpha\beta}\, dt^{\wedge} + \chi_{\alpha}\, d\pi^{\alpha} + \varkappa_{\alpha}\, dm^{\alpha} + \theta_{\alpha}{}^{\beta} d\nabla_{\beta}\pi^{\alpha} + \\ + \vartheta_{\alpha}{}^{\beta} d\nabla_{\beta} m^{\alpha} + \Phi^{\beta\alpha\gamma} d\frac{\partial g_{\alpha\gamma}}{\partial\xi^{\beta}} - \frac{1}{\rho}\nabla_{\beta}\Omega^{\beta}\, dt^{\wedge} + \frac{\mathrm{E}}{\rho}\cdot \mathbf{j}\, dt^{\wedge} + \frac{d^{*}q}{\rho} - T\, dS = 0\end{aligned} \tag{40}$$

where φ, $\psi^{\alpha\beta}$, $\Omega^{\alpha\beta}$, χ_{α}, $\varkappa^{\alpha}$, $\theta^{\cdot\beta}_{\alpha\cdot}$, $\vartheta^{\cdot\beta}_{\alpha\cdot}$ and $\Phi^{\alpha\beta\gamma}$ are defined by Formulas

(41)

$$-S = \frac{\partial F}{\partial T} + \varphi, \qquad \frac{p^{\alpha\beta} + p^{\beta\alpha}}{2} = 2\rho\,\frac{\partial F}{\partial g_{\alpha\beta}} - \frac{2}{\sqrt{g^{*}}}\frac{\partial}{\partial\xi^{\lambda}}\left[\rho\sqrt{g^{*}}\,\frac{\partial F}{\partial\left(\partial g_{\alpha\beta}/\partial\xi^{\lambda}\right)}\right] +$$

$$+ \frac{1}{2}\left(E_{\gamma}P^{\gamma} + B_{\gamma}M^{\gamma}\right) g^{*\alpha\beta} - \frac{1}{2}\left(E^{\alpha}P^{\beta} + E^{\beta}P^{\alpha} + B^{\alpha}M^{\beta} + B^{\beta}M^{\alpha}\right) +$$

$$+ \frac{1}{2}\nabla_{\lambda}\left[(R^{\beta\lambda} - R^{\lambda\beta})\pi^{\alpha} + (R^{\alpha\lambda} - R^{\lambda\alpha})\pi^{\beta} + (R^{\beta\alpha} + R^{\alpha\beta})\pi^{\lambda}\right] +$$

$$+ \frac{1}{2}\nabla_{\lambda}\left[(N^{\beta\lambda} - N^{\lambda\beta}) m^{\alpha} + (N^{\alpha\lambda} - N^{\lambda\alpha}) m^{\beta} + (N^{\beta\alpha} + N^{\alpha\beta}) m^{\lambda}\right] -$$

---

*) In what follows, the components of all vectors and tensors are taken in a co-moving system of coordinates. For simplicity, the symbol ^ , denoting components in the co-moving system, is dropped. Further arguments and equations will be simplified if, instead of the system of defining parameters (37), we take the system $T,\ g^{\circ}_{ij},\ g_{ij},\ \pi^{\alpha},\ m^{\alpha},\ \nabla^{\circ}{}_{\beta}\pi^{\alpha},\ \nabla^{\circ}{}_{\beta}m^{\alpha},\ \nabla^{\circ}{}_{\beta}g_{\alpha\gamma}$

Here, $\overset{\circ}{\nabla}_{\beta}$ is the symbol for the covariant derivative in the three-dimensional space of the initial state. Below, the case of saturated magnetization, for which $|\mathbf{m}|$ = const , is not considered. Inclusion of saturation will not introduce essential difficulties.

$$- \frac{2}{\sqrt{g^*}} \frac{\partial \rho \sqrt{g^*} \Phi^{\lambda\alpha\beta}}{\partial \xi^\lambda} + \rho \psi^{\alpha\beta} \qquad (41) \text{ cont.}$$

$$p^{\alpha\beta} - p^{\beta\alpha} + E^\alpha P^\beta - E^\beta P^\alpha + B^\alpha M^\beta - B^\beta M^\alpha = 2\rho \Omega^{\alpha\beta}$$

$$E_\alpha = \frac{\partial F}{\partial \pi^\alpha} + \frac{1}{\rho} \nabla_\beta R_{\alpha \cdot}^{\cdot\beta} + \chi_\alpha, \quad R_{\alpha \cdot}^{\cdot\beta} = -\rho \frac{\partial F}{\partial \nabla_\beta \pi^\alpha} + \rho \theta_{\alpha \cdot}^{\cdot\beta}$$

$$B_\alpha = \frac{\partial F}{\partial m^\alpha} + \frac{1}{\rho} \nabla_\beta N_{\alpha \cdot}^{\cdot\beta} + \varkappa_\alpha, \quad N_{\alpha \cdot}^{\cdot\beta} = -\rho \frac{\partial F}{\partial \nabla_\beta m^\alpha} + \rho \vartheta_{\alpha \cdot}^{\cdot\beta}$$

$$\Lambda^{\beta\alpha\gamma} = -\rho \frac{\partial F}{\partial (\partial g_{\alpha\gamma} / \partial \xi^\beta)} + \frac{[(R^{\gamma\beta} - R^{\beta\gamma}) \pi^\alpha + (R^{\alpha\beta} - R^{\beta\alpha}) \pi^\gamma + (R^{\gamma\alpha} + R^{\alpha\gamma}) \pi^\beta}{4} +$$

$$+ \frac{[(N^{\gamma\beta} - N^{\beta\gamma}) m^\alpha + (N^{\alpha\beta} - N^{\beta\alpha}) m^\gamma + (N^{\gamma\alpha} + N^{\alpha\gamma}) m^\beta]}{4} - \rho \Phi^{\beta\alpha\gamma}$$

The tensor $\psi^{\alpha\beta}$ is symmetric and the tensor $\Omega^{\alpha\beta}$ is antisymmetric. The components $\Lambda^{\beta\alpha\gamma}$ and $\Phi^{\beta\alpha\gamma}$ are symmetric with respect to the last two indices.

If it is assumed that the inflows of energy $-\rho^{-1} \nabla_\beta \Omega^\beta dt^\wedge$ and $\rho^{-1} d^* q$ correspond to the inflow of heat energy, then for reversible and for certain irreversible processes (for example, with heat conductivity and radiation included), Equation

$$T dS = \frac{1}{\rho} \mathbf{E} \cdot \mathbf{j} \, dt + \frac{1}{\rho} d^* q - \frac{1}{\rho} \nabla_\beta \Omega^\beta dt = dq^{(e)} \qquad (42)$$

will be satisfied.

If, moreover, it is admitted that the quantities $\varphi, \psi^{\alpha\beta}, \Omega^{\alpha\beta}, \chi_\alpha, \varkappa_\alpha, \theta_\alpha{}^\beta$, $\vartheta_\alpha{}^\beta$ and $\Phi^{\beta\alpha\gamma}$, defined by Equations (41), do not depend on derivatives with respect to time (*) or on the defining parameters (37), then, from Equation (40), and in view of the fact that the increments with respect to time are independent of the defining parameters (**), we obtain

$$\varphi = \psi^{\alpha\beta} = \Omega^{\alpha\beta} = \chi_\alpha = \varkappa_\alpha = \theta_\alpha{}^\beta = \vartheta_\alpha{}^\beta = \Phi^{\beta\alpha\gamma} = 0 \qquad (43)$$

Thus, on the basis of (42) and (43), we find that Equations (41) define the equations of state for the material medium. The equations are the generalization of the ordinary equations of the theory of elasticity for the case where the free energy depends on gradients of the polarization vector, the magnetization vector, and gradients of the deformation tensor.

If F depends only on T, $g_{\hat{\alpha}\beta}$, $\overset{\circ}{g}_{\alpha\beta}$, $\pi^{\wedge\alpha}$ and $m^{\wedge\alpha}$, and does not depend on their gradients, then $R_\alpha{}^\beta = N_\alpha{}^\beta = \Lambda^{\beta\gamma\alpha} = 0$.

In this case, the components of the vector $Q dt^\wedge$, $Q^\beta dt^\wedge = \Omega^\beta dt^\wedge$ determine the influx of heat, while Equations (41) reduce to the equations of state of the theory of elasticity, including electrical polarization and magnetization.

---

*) Only the assumption of independence of derivatives with respect to time is essential. Hypotheses about the dependence or independence of these coefficients on any derivatives with respect to coordinates is not necessary.

**) We can construct models in which derivatives with respect to time may be linearly dependent on the defining parameters [7].

For elucidating gyromagnetic effects, it is necessary to take into account the dependence of the free energy of the vorticity vector $\boldsymbol{\omega} = {}^1/_2 \operatorname{rot} \mathbf{v}$. If the list of defining parameters (37) is supplemented by the components of the axial vorticity vector ω^γ, (*), connected with the antisymmetric tensor $\omega_{\alpha\beta}$, then there appears in the left-hand side of (40) a term of the form $-\partial F / \partial\omega^\gamma$. When ω^γ is included in the system of defining parameters (37), the components ω^γ and $d\omega^\gamma / dt^\wedge$ for the multitude of possible processes may be regarded in Equation (40) as quantities which are independent of the other parameters in system (37) and of their derivatives with respect to time. From this it follows that $\partial F / \partial\omega^\gamma = 0$ and $(\partial F / \partial\omega^\gamma) d\omega^\gamma = 0$, since otherwise the equations $\partial F / \partial\omega^\gamma = 0$ would represent universal relations between the defining parameters. On the other hand, if relation (42) is retained, as well as the other hypotheses about the independence of all coefficients in Equation (40) from increments in the system of parameters (37), and also from ω^γ and $d\omega^\gamma/dt$, then, together with Equations (43), we again obtain the result

$$\frac{\partial F}{\partial\omega^\gamma} d\omega^\gamma = 0$$

which contradicts the statement of the problem, and, therefore, the basic hypotheses must be altered in this case.

In connection with this, let us investigate an example of the generalization of the foregoing theory, based on the following very weak hypotheses (the resulting relations and conclusions are also applicable if F does not depend on ω^γ).

1. To take into account the irreversible nature of magnetization, we replace (42) by

$$TdS = \frac{1}{\rho}\mathbf{E}\cdot\mathbf{j}dt + \frac{1}{\rho} d^*q - \frac{1}{\rho}\nabla_\beta\Omega^\beta dt + dq' = dq^{(e)} + dq' \quad \left(dq' = c_{\alpha\beta}\frac{dm^\alpha}{dt}\frac{dm^\beta}{dt}\right) \tag{44}$$

where $c_{\alpha\beta}$ are components in a co-moving coordinate system of a symmetric tensor which depends in general on the defining parameters (37).

2. In Equation (40), which acquires the form

$$\begin{aligned}&\varphi dT + \psi^{\alpha\beta} e_{\alpha\beta} dt + D_\gamma\omega^\gamma dt + \chi_\alpha d\pi^\alpha + \varkappa_\alpha dm^\alpha + \theta_{\alpha\cdot}^{\cdot\beta} d\nabla_\beta\pi^\alpha + \\ &+ \vartheta_{\alpha\cdot}^{\cdot\beta} d\nabla_\beta m^\alpha + \Phi^{\beta\alpha\gamma} d\frac{\partial g_{\alpha\gamma}}{\partial\xi^\beta} - \frac{\partial F}{\partial\omega^\gamma} d\omega^\gamma - c_{\alpha\beta}\frac{dm^\alpha}{dt}\frac{dm^\beta}{dt} = 0\end{aligned} \tag{45}$$

we assume that the coefficients of $\omega^\gamma dt$ and of linearly independent increments of the defining parameters may depend on the defining parameters and on the following derivatives with respect to time, $d\omega^\gamma / dt$, $d\pi^\alpha / dt$ and dm^α/dt. (In Equation (45) $D_\gamma = -2\Omega^{\alpha\beta}$, $\omega^\gamma = -\omega_{\alpha\beta}$, where α, β, γ form a cyclic permutation of the indices 1, 2, 3).

From 1, 2, and Equation (45), it follows that (**)

$$\varphi = \psi^{\alpha\beta} = \theta_\alpha^{\ \beta} = \vartheta_\alpha^{\ \beta} = \Phi^{\beta\alpha\gamma} = 0, \qquad \mathbf{D}\cdot\boldsymbol{\omega} + \boldsymbol{\chi}\cdot\frac{d\boldsymbol{\pi}}{dt} + (\boldsymbol{\varkappa} - \mathbf{C})\cdot\frac{d\mathbf{m}}{dt} - \frac{\partial F}{\partial\omega^\gamma}\frac{d\omega^\gamma}{dt} = 0 \tag{46}$$

where

$$\boldsymbol{\chi} = \chi_\alpha \mathbf{\ni}^{\alpha\wedge}, \qquad \boldsymbol{\varkappa} = \varkappa_\alpha \mathbf{\ni}^{\alpha\wedge}, \qquad \mathbf{C} = c_{\alpha\beta}\frac{dm^\beta}{dt}\mathbf{\ni}^{\wedge\alpha}, \qquad \boldsymbol{\omega} = \omega^\gamma \mathbf{\ni}_\gamma^{\ \wedge}$$

$$\mathbf{D} = D_\gamma \mathbf{\ni}^{\wedge\gamma} = \frac{p^{\beta\alpha} - p^{\alpha\beta}}{\rho}\mathbf{\ni}^{\wedge\gamma} + \boldsymbol{\pi}\times\boldsymbol{E} + \mathbf{m}\times\mathbf{B} \tag{47}$$

---

*) For what follows, it is essential that in the several arguments of the free energy F the components ω^γ are included and the tensor components of the gradient of the vorticity vector, $\nabla_\alpha\omega^\gamma$., are not included. Due to this hypothesis, and the hypothesis of linear independence from time derivatives, terms of the form $M_\gamma^{\ \alpha} d\omega^\gamma$ do not appear in the right-hand side of (38).

**) Sedov, L.I., Certain problems in the construction of new models of continuous media. Contribution to XV Int.Congr.theor. and appl.Mech., Munich, 1964.

in which time derivatives of vectors are taken with respect to a co-moving coordinate system (for $\ni_{\alpha}{}^{\wedge}$ = const).

From assumptions made, it follows that Equation (46) is equivalent to the following three vector equations

$$\mathrm{D} = k_1\omega + \omega \times \mathrm{G}_1 \tag{48}$$

$$\chi = k_2 \frac{d\pi}{dt} + \frac{d\pi}{dt} \times \mathrm{G}_2 \tag{49}$$

$$\frac{dm}{dt} = k_3 (\varkappa - \mathrm{C}) + (\varkappa - \mathrm{C}) \times \mathrm{G}_3 \tag{50}$$

where $\mathrm{G}_1, \mathrm{G}_2, \mathrm{G}_3$ are arbitrary vectors, and the scalars k_1, k_2 and k_3 are connected by a single relation

$$k_1 |\omega|^2 + k_2 \left|\frac{d\pi}{dt}\right|^2 + k_3 |\varkappa - \mathrm{C}|^2 = \frac{\partial F}{\partial \omega^{\gamma}} \frac{d\omega^{\gamma}}{dt} \tag{51}$$

To eliminate arbitrariness from relations (48) to (50), it is necessary to fall back on supplementary hypotheses of physical nature.

For example, it may be assumed that Equation (48) is the equation of moments of momentum for the material medium, and that the right-hand side is equal to the time derivative relative to an inertial system referenced to the internal moment of momenta $\mathbf{K}$ per unit mass. As is well known [8], it may be assumed that $\mathbf{K} = -\gamma \mathbf{m}$, where γ is a known constant. With this hypothesis, the scalar k_1 and the essential part of the vector $\mathbf{G}_1$ are fixed. The Equation (48) may be regarded as the definition of the nonsymmetric part of the stress tensor. The polarization equation (49) can be fixed if it is assumed that the electric intensity vector $\mathbf{E}$ is determined by the free energy, depending on the system (37) through quasi-static relations; it then follows that $\chi_{\alpha} = 0$, and therefore $k_2 = 0$, and it may be assumed that $\mathbf{G}_2 = 0$.

After these hypotheses, the scalar k_3 is determined by Equation (51), while Equation (50), after determination of vector $\mathbf{G}_3$, may be regarded as a possible alteration or a certain generalization of the phenomenological equation of Landau and Lifshits [1]. This equation was proposed by them for the theory of magnetic waves in ferromagnetics, with accelerations and deformations not taken into account.

In the paper of Vlasov and Ishmukhatov and those of a number of other authors cited in [9], variational principles, introduction of hypothetical, appropriate Lagrangian functions, and certain supplementary assumptions, are used to obtain various systems of kinetic equations and equations of state, including deformations of the medium.

Further development of the present theory, which is based on the equation of heat flux (40), to the case of models of media with irreversible processes of a more general kind (including viscosity, temperature gradients and various effects) can be carried out in an analogous way with the help of macroscopic theories of the Onsager type.

BIBLIOGRAPHY

1. Landau, L.D. and Lifshits, E.M., Electrodinamika sploshnykh sred (Electrodynamics of Continuous Media). Fizmatgiz, M., 1957 (*).

2. Tonnelat, Marie-Antoinette, Osnovy elektromagnetizma i teorii otnositel'nosti (The Principles of Electromagnetism and of Relativity). Izd. inostr.lit., M., 1962 (**).

3. Synge, L.L., Relativity, the Special Theory. North-Holland Publ.Comp., Amsterdam, 1956.

4. Pauli, W., Teoriia otnositel'nosti (The Theory of Relativity). Gostekhizdat, M., 1947 (***).

5. De Groot, S.R and Mazur, P., Neravnovesnaia termodinamika (Nonequilibrium Thermodynamics). Publ. "Mir", M., 1964 (****).

6. Sedov, L.I., Vvedenie v mekhaniku sploshnykh sred (Introduction to the Mechanics of Continuous Media). Fizmatgiz, 1962.

7. Sedov, L.I. and Eglit, M.E., Postroenie negolonomnyk modelei sploshnykh sred s uchetom konechnosti deformatsii i nekotorykh fiziko-khimicheskikh effektov (Construction of nonholonomic models of continuous media, including finite deformations and certain physico-chemical effects). Dokl.Akad.Nauk SSSR, Vol.142, № 1, 1962

8. Gurevich, A.G., Ferrity na sverkhvysokikh chastotakh (Ferrites at Superhigh Frequencies). Fizmatgiz, 1960.

9. Vlasov, K.B. and Ishmukhatov, B.Kh., Uravneniia dvizheniia i sostoianiia magnitouprugikh sred (Equations of motion and state of magnetoelastic media). ZhETF, Vol.64, № 1, 1964.

Translated by A.R.

EDITORIAL NOTE.

*) Addison-Wesley Publ., 1960.

**) Transl. from the French, D.Reidl Publ.Co., Holland, 1964.

***) Encyclopedia der Math.Wissenschaften, Band V.2, Heft 4, B.G. Teubner, Leipzig, 1921.

****) North-Holland Publishing Co., Amsterdam, 1962.